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Groups of measure-preserving homeomorphisms of noncompact 2-manifolds

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Abstract

Suppose M is a noncompact connected 2-manifold and μ is a good Radon measure of M with $\mu(\partial M) = 0$. Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M equipped with the compact-open topology and $\mathcal{H}(M)_0$ denote the identity component of $\mathcal{H}(M)$. Let $\mathcal{H}(M;\mu)$ denote the subgroup of $\mathcal{H}(M)$ consisting of μ-preserving homeomorphisms of M and $\mathcal{H}(M;\mu)_0$ denote the identity component of $\mathcal{H}(M;\mu)$. We use results of A. Fathi and R. Berlanga to show that $\mathcal{H}(M;\mu)_0$ is a strong deformation retract of $\mathcal{H}(M)_0$ and classify the topological type of $\mathcal{H}(M;\mu)_0$.

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1. Introduction

The purpose of this article is to study topological properties of the groups of measure-preserving homeomorphisms of noncompact 2-manifolds. Suppose M is a connected 2-manifold and X is a compact subpolyhedron of M with respect to some triangulation of M. Let $\mathcal{H}_X(M)$ denote the group of homeomorphisms h of M such that $h|_X = \mathrm{id}_X$, equipped with the compact-open topology, and let $\mathcal{H}_X(M)_0$ denote the *connected* component of id_M in $\mathcal{H}_X(M)$. Suppose μ is a good Radon measure on M such that $\mu(\mathrm{Fr}\, X \cup \partial M) = 0$ (cf. Section 3). Let $\mathcal{H}_X(M;\mu)$ denote the subgroup of $\mathcal{H}_X(M)$ consisting of μ-preserving homeomorphisms and let $\mathcal{H}_X(M,\mu)_0$ denote the *connected* component of id_M in $\mathcal{H}_X(M,\mu)$.

A. Fathi and R. Berlanga introduced an intermediate subgroup $\mathcal{H}_X(M,\mu\text{-end-reg})$ between $\mathcal{H}_X(M)$ and $\mathcal{H}_X(M,\mu)$. According to R. Berlanga [3] $h \in \mathcal{H}(M)$ is said to be μ-end-regular if h preserves μ-null sets and μ-finite ends (see Section 3). Let $\mathcal{H}_X(M,\mu\text{-end-reg})$ denote the subgroup of $\mathcal{H}_X(M)$ consisting of μ-end-regular homeomorphisms of M and let $\mathcal{H}_X(M,\mu\text{-end-reg})_0$ denote the connected component of id_M in $\mathcal{H}_X(M,\mu\text{-end-reg})$.

When M is compact, $\mathcal{H}_X(M)$ is an ANR [10] (cf. [15]) and A. Fathi [5] showed that $\mathcal{H}(M,\mu)$ is a strong deformation retract of $\mathcal{H}(M,\mu\text{-end-reg})$ and the latter is homotopy dense in $\mathcal{H}(M)$. This implies that $\mathcal{H}(M,\mu)$ is an ANR

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and a strong deformation retract of $\mathcal{H}(M)$. The topological characterization of ℓ_2-manifolds [4] implies that $\mathcal{H}(M,\mu)$ is a ℓ_2-manifold.

To the case where M is noncompact, R. Berlanga [1–3] extended the section theorem for the action of $\mathcal{H}(M)$ on the space of good Radon measures on M [11,5], and showed that $\mathcal{H}(M,\mu)$ is a strong deformation retract of $\mathcal{H}(M,\mu\text{-end-reg})$. On the other hand, we have shown that $\mathcal{H}_X(M)_0$ is an ANR [16] and $\mathcal{H}_X^{\mathrm{PL}}(M)_0$ is homotopy dense in $\mathcal{H}_X(M)_0$ [17]. Here $\mathcal{H}_X^{\mathrm{PL}}(M)_0$ is the connected component of id_M in the group of PL-homeomorphisms of M (with respect to any triangulation of M). Since we can isotope the triangulation of M so that $\mathcal{H}_X^{\mathrm{PL}}(M)_0 \subset \mathcal{H}_X(M,\mu\text{-end-reg})_0$ (Section 4), it follows that $\mathcal{H}_X(M,\mu\text{-end-reg})_0$ is also homotopy dense in $\mathcal{H}_X(M)_0$. Some sort of arguments on triangulation is necessary to include the compact polyhedron X in our statements. We can combine these results together to obtain the noncompact version of Fathi's results in dimension 2.

Theorem 1.1. *Suppose M is a connected 2-manifold, X is a compact subpolyhedron of M with respect to some triangulation of M and μ is a good Radon measure on M such that $\mu(\mathrm{Fr}_M X \cup \partial M) = 0$. Then $\mathcal{H}_X(M,\mu)_0$ is an ANR and it is a strong deformation retract of $\mathcal{H}_X(M)_0$.*

The homotopy type of $\mathcal{H}_X(M)_0$ has been classified in [7,16]. It turns out that $\mathcal{H}_X(M)_0$ has the homotopy type of a compact polyhedron P, which is a point, a circle, a torus or $SO(3)$ as described in [7,16] (cf. [13]). By Theorem 1.1 $\mathcal{H}_X(M,\mu)_0$ has the same homotopy type as $\mathcal{H}_X(M)_0$ and the infinite-dimensional manifold theory (cf. [14]) enables us to classify the topological type of $\mathcal{H}_X(M,\mu)_0$.

Corollary 1.1. *If $X \neq M$, then*

(i) *$\mathcal{H}_X(M,\mu)_0$ is a topological ℓ_2-manifold, and*
(ii) *$\mathcal{H}_X(M,\mu)_0 \cong P \times \ell_2$, where P is a compact polyhedron homotopy equivalent to $\mathcal{H}_X(M,\mu)_0$.*

This paper is organized as follows. Section 2 is devoted to generalities on ANR's, ℓ_2-manifolds, homeomorphism groups and ends of spaces. Section 3 includes fundamental facts on spaces of Radon measures. In Section 4 we show some properties of Radon measures which are necessary to prove Theorem 1.1.

2. Homeomorphism groups of noncompact 2-manifolds

2.1. Conventions

Throughout the paper spaces are assumed to be separable and metrizable, and maps are always continuous (otherwise specified). The symbol $\cong$ indicates a homeomorphism and $\simeq$ denotes a homotopy equivalence (HE). The term "strong deformation retract (or retraction)" is abbreviated as SDR. When A is a subset of a space X, the symbols $\mathrm{Fr}_X A$, $\mathrm{cl}_X A$ and $\mathrm{Int}_X A$ denote the frontier, closure and interior of A relative to X. When M is a manifold, $\partial = \partial M$ and $\mathrm{Int}\, M$ denote the boundary and interior of M as a manifold.

2.2. ANR's and ℓ_2-manifolds

A metrizable space X is called an *ANR* (*absolute neighborhood retract*) if any map $f : B \to X$ from a closed subset B of a metrizable space Y has an extension to a neighborhood U of B [9].

Definition 2.1. A subspace B of a space Y is said to be *homotopy dense* (HD) in Y (or B has the homotopy absorption property in Y) if there exists a homotopy $f_t : Y \to Y$ $(0 \leqslant t \leqslant 1)$ such that $f_0 = \mathrm{id}_Y$ and $f_t(Y) \subset B$ $(0 < t \leqslant 1)$.

Lemma 2.1. *If B is HD in Y, then*

(i) *the inclusion $B \subset Y$ is a HE, and*
(ii) *Y is an ANR iff B is an ANR* [8].

The symbol ℓ_2 denotes the separable Hilbert space $\{(x_n) \in \mathbb{R}^\infty \colon \sum_n x_n^2 < \infty\}$. An *$\ell_2$-manifold* is a separable metrizable space which is locally homeomorphic to ℓ_2. For topological groups there is a simple characterization of ℓ_2-manifolds.

Theorem 2.1. (T. Dobrowolski and H. Toruńczyk [4]) *A topological group G is an ℓ_2-manifold iff it is a separable, non-locally compact, completely metrizable ANR.*

2.3. Homeomorphism groups of noncompact 2-manifolds

Suppose Y is a locally connected, locally compact, separable metrizable space and A, X are closed subsets of Y. Let $\mathcal{H}_X(Y, A)$ denote the group of homeomorphisms h of Y such that $h(A) = A$ and $h|_X = \mathrm{id}_X$, equipped with the compact-open topology. $\mathcal{H}_X(Y, A)_0$ denotes the connected component of id_Y in $\mathcal{H}_X(Y, A)$. It is known that $\mathcal{H}_X(Y, A)$ is a separable, completely metrizable, topological group and $\mathcal{H}_X(Y, A)_0$ is a closed subgroup of $\mathcal{H}_X(Y, A)$. In general, for any topological group G, the connected component G_0 of the unit element 1_G in G is a closed subgroup of G (cf. [12]), and if G is locally path-connected, then G_0 coincides with the path-component of 1_G in G.

Definition 2.2. When Y is a polyhedron, $\mathcal{H}_X^{\mathrm{PL}}(Y, A)$ denotes the subgroup of $\mathcal{H}_X(Y, A)$ consisting of PL-homeomorphisms of Y and $\mathcal{H}_X^{\mathrm{PL}}(Y, A)_0$ denotes the connected component of id_Y in $\mathcal{H}_X^{\mathrm{PL}}(Y, A)$.

Every 2-manifold has a PL-structure.

Theorem 2.2. *Suppose M is a connected PL 2-manifold and X is a compact subpolyhedron of M. Then*

(i) $\mathcal{H}_X(M)_0$ *is an ANR* [10,16], *and*
(ii) $\mathcal{H}_X^{\mathrm{PL}}(M)_0$ *is HD in* $\mathcal{H}_X(M)_0$ [6,17].

2.4. Ends of spaces (cf. [3])

Suppose Y is a connected, locally connected, locally compact, separable metrizable space. Let $\mathcal{K}(Y)$ denote the set of compact subsets of Y and for each $K \in \mathcal{K}(Y)$ let $\mathcal{C}(Y - K)$ denote the set of connected components of $Y - K$.

Definition 2.3.

(i) An *end* of Y is a function e which assigns an $e(K) \in \mathcal{C}(Y - K)$ to each $K \in \mathcal{K}(Y)$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$.
(ii) The symbol $\mathcal{E}(Y)$ denotes the set of ends of Y.
(iii) The *end compactification* of Y is the space $\overline{Y} = Y \cup \mathcal{E}(Y)$ equipped with the topology defined by the following conditions:
  (a) Y is an open subspace of $\overline{Y}$,
  (b) the fundamental open neighborhoods of $e \in \mathcal{E}(Y)$ is given by
$$N(e, K) = e(K) \cup \{e' \in \mathcal{E}(Y) \mid e'(K) = e(K)\} \quad (K \in \mathcal{K}(Y)).$$
(iv) The set of ends, $\mathcal{E}(Y)$, is assigned the subspace topology of $\overline{Y}$.

The space $\overline{Y}$ is compact, connected, metrizable and Y is a dense open subset of $\overline{Y}$, while the remainder $\mathcal{E}(Y)$ is compact and 0-dimensional. (If Y is compact, then $\mathcal{E}(Y) = \emptyset$ and $\overline{Y} = Y$.)

Suppose Y and Z are connected, locally connected, locally compact separable metric spaces and $f : Y \to Z$ is a *proper* map ($f^{-1}(K)$ is compact for any $K \in \mathcal{K}(Z)$). For each $e \in \mathcal{E}(Y)$ we define an end $f(e) \in \mathcal{E}(Z)$ as follows: For any $K \in \mathcal{K}(Z)$ we have $f^{-1}(K) \in \mathcal{K}(Y)$ and $e(f^{-1}(K)) \in \mathcal{C}(Y - f^{-1}(K))$. Since $f(e(f^{-1}(K)))$ is a connected subset of $Z - K$, there exists a unique $U \in \mathcal{C}(Z - K)$ with $f(e(f^{-1}(K))) \subset U$. We put $f(e)(K) = U$. The map f has a natural extension $\overline{f} : \overline{Y} \to \overline{Z}$ defined by $\overline{f}(e) = f(e)$ $(e \in \mathcal{E}(Y))$. In particular, for $h \in \mathcal{H}(Y)$ and $e \in \mathcal{E}(Y)$ the end $h(e) \in \mathcal{E}(Y)$ is defined by $h(e)(K) = h(e(h^{-1}(K)))$ $(K \in \mathcal{K}(Y))$. It follows that $\overline{h} \in \mathcal{H}(\overline{Y})$ and that if $h \in \mathcal{H}(Y)_0$ then $h(e) = e$ $(e \in \mathcal{E}(Y))$, since $\mathcal{E}(Y)$ is totally disconnected (each connected component consists of a single point).

3. Fundamental facts on Radon measures

3.1. Spaces of Radon measures

Next we recall general facts on spaces of Radon measures (cf. [3,5]). Suppose Y is a connected, locally connected, locally compact, separable metrizable space. Let $\mathcal{B}(Y)$ denote the σ-algebra of Borel subsets of Y and let $C_0(Y)$ denote the linear space of continuous functions $f : Y \to \mathbb{R}$ with compact support. A *Radon measure* on Y is a measure μ on the measurable space $(Y, \mathcal{B}(Y))$ such that $\mu(K) < \infty$ for any compact subset K of Y. Let $\mathcal{M}(Y)$ denote the set of Radon measures on Y. Each $\mu \in \mathcal{M}(Y)$ induces a positive continuous linear functional $\varphi_\mu : C_0(Y) \to \mathbb{R}$: $\varphi_\mu(f) = \int_Y f\,\mathrm{d}\mu$. (The positivity of φ_μ means that $\varphi_\mu(f) \geqslant 0$ for $f \geqslant 0$.) This yields a 1–1 correspondence between the space $\mathcal{M}(Y)$ and the space of positive continuous linear functionals $\varphi : C_0(Y) \to \mathbb{R}$. This correspondence motivates the following definition.

Definition 3.1. The *weak* topology w on $\mathcal{M}(Y)$ is the weakest topology such that the function

$$\Phi_f : \mathcal{M}(Y) \to \mathbb{R}:\ \Phi_f(\mu) = \int_Y f\,\mathrm{d}\mu$$

is continuous for any $f \in C_0(Y)$. The notation $\mathcal{M}(Y)_w$ denotes the space $\mathcal{M}(Y)$ equipped with the weak topology w.

For $\mu \in \mathcal{M}(Y)$ and $A \in \mathcal{B}(Y)$ the restriction $\mu|_A \in \mathcal{M}(A)$ is defined by $(\mu|_A)(B) = \mu(B)$ $(B \in \mathcal{B}(A))$.

Lemma 3.1. [3, Lemma 2.2] *For any closed subset A of Y the function*

$$\mathcal{M}(Y)_w \to \mathcal{M}(A)_w:\ \mu \mapsto \mu|_A$$

is continuous at each $\mu \in \mathcal{M}(Y)$ with $\mu(\mathrm{Fr}\,A) = 0$.

We say that $\mu \in \mathcal{M}(Y)$ is *good* if $\mu(p) = 0$ for any point $p \in Y$ and $\mu(U) > 0$ for any nonempty open subset U of Y. For $A \in \mathcal{B}(Y)$ let $\mathcal{M}_g^A(Y)$ denote the subset of good Radon measures μ on Y with $\mu(A) = 0$.

Definition 3.2. For $\mu \in \mathcal{M}(Y)$ the function $\alpha(\mu) : \mathcal{E}(Y) \to \{0, \infty\}$ is defined by

$$\alpha(\mu)(e) = \begin{cases} 0 & \mu(e(K)) < \infty \text{ for some } K \in \mathcal{K}(Y)), \\ \infty & (\mu(e(K)) = \infty \text{ for any } K \in \mathcal{K}(Y)). \end{cases}$$

We obtain the subspaces of *μ-finite ends* and *μ-infinite ends*,

$$\mathcal{E}_f(Y;\mu) = \{e \in \mathcal{E}(Y) \mid \alpha(\mu)(e) = 0\} \quad \text{and}$$
$$\mathcal{E}_i(Y;\mu) = \{e \in \mathcal{E}(Y) \mid \alpha(\mu)(e) = \infty\}.$$

Definition 3.3. For $A, X \in \mathcal{B}(Y)$ and $\mu \in \mathcal{M}_g^A(Y)$ we consider the following subspaces of $\mathcal{M}_g^A(Y)$:

(i) $\mathcal{M}_g^A(Y; \mu\text{-end-reg}) = \{\nu \in \mathcal{M}_g^A(Y) \mid (a), (b), (c)\}$:
  (a) $\nu(Y) = \mu(Y)$,
  (b) ν has the same null sets as μ (i.e., $\nu(B) = 0$ iff $\mu(B) = 0$ for any $B \in \mathcal{B}(Y)$),
  (c) $\alpha(\nu) = \alpha(\mu)$.

(ii) $\mathcal{M}_g^A(Y, X; \mu\text{-end-reg}) = \{\nu \in \mathcal{M}_g^A(Y; \mu\text{-end-reg}) \mid (d), (e)\}$:
  (d) $\nu|_X = \mu|_X$,
  (e) $\nu(C) = \mu(C)$ for any $C \in \mathcal{C}(Y - X)$.

Suppose $\mu \in \mathcal{M}_g(Y)$. We consider the subspace $Y \cup \mathcal{E}_f(Y;\mu)$ of $\overline{Y}$ and the space $\mathcal{M}(Y \cup \mathcal{E}_f(Y;\mu))_w$ of Radon measures on $Y \cup \mathcal{E}_f(Y;\mu)$. Each $\nu \in \mathcal{M}_g(Y; \mu\text{-end-reg})$ has a natural extension $\bar{\nu} \in \mathcal{M}_g(Y \cup \mathcal{E}_f(Y;\mu))$ defined by $\bar{\nu}(B) = \nu(B \cap Y)$ $(B \in \mathcal{B}(Y \cup \mathcal{E}_f(Y;\mu)))$.

Definition 3.4. [3, Section 3, p. 245] The *finite-end weak* topology ew on $\mathcal{M}_g(Y;\mu\text{-end-reg})$ is the weakest topology for which the following injection is continuous:

$$\iota:\mathcal{M}_g(Y;\mu\text{-end-reg}) \to \mathcal{M}\big(Y \cup \mathcal{E}_f(Y;\mu)\big)_w:\ \nu \mapsto \bar{\nu}.$$

The notation $\mathcal{M}_g(Y;\mu\text{-end-reg})_{ew}$ denotes the space $\mathcal{M}_g(Y;\mu\text{-end-reg})$ equipped with the topology ew.

The space $\mathcal{M}_g(Y;\mu\text{-end-reg})_{ew}$ admits a canonical contraction

$$\varphi_t(\nu) = (1-t)\nu + t\mu \quad (0 \leqslant t \leqslant 1).$$

For any $A, X \in \mathcal{B}(Y)$ the contraction φ_t maps the subspace $\mathcal{M}_g^A(Y, X;\mu\text{-end-reg})_{ew}$ into itself and induces a contraction of this subspace.

Lemma 3.2. *Suppose $\mu \in \mathcal{M}_g(Y)$, X is a compact subset of Y with $\mu(\mathrm{Fr}_Y X) = 0$, $U \in \mathcal{C}(Y - X)$ and $A = \mathrm{cl}_Y U$. Assume that A is locally connected. Then the following restriction map is continuous*:

$$r:\mathcal{M}_g(Y, X;\mu\text{-}\textit{end-reg})_{ew} \to \mathcal{M}_g(A;\mu|_A\text{-}\textit{end-reg})_{ew}, \quad r(\nu) = \nu|_A.$$

Proof. We use the following notations: $\partial U = \{e \in \mathcal{E}(Y) \mid e(X) = U\}$, $Y_1 = Y \cup \mathcal{E}_f(Y;\mu)$ and $A_1 = A \cup (\partial U \cap \mathcal{E}_f(Y;\mu))$. Then A_1 is a closed subset of Y_1 and the inclusion $i : A \subset Y$ induces a homeomorphism $\bar{i} : A \cup \mathcal{E}_f(A;\mu|_A) \cong A_1$.

Consider the following commutative diagram :

$$\begin{tikzcd}
\mathcal{M}_g(Y, X;\mu\text{-end-reg})_{ew} \arrow[rr, "r"] \arrow[d, hook, "\iota"] & & \mathcal{M}_g(A;\mu|_A\text{-end-reg})_{ew} \arrow[d, hook, "\iota_A"] \\
\mathcal{M}_g^{\mathrm{Fr}_{Y_1} A_1}(Y_1)_w \arrow[r, "r_1"'] & \mathcal{M}_g(A_1)_w & \mathcal{M}_g(A \cup \mathcal{E}_f(A;\mu|_A))_w \arrow[l, "\cong", "\bar{i}_*"']
\end{tikzcd}$$

Here, $r(\nu) = \nu|_A$, $\iota(\nu) = \bar{\nu}$, $\iota_A(\lambda) = \bar{\lambda}$, $r_1(\lambda) = \lambda|_{A_1}$ and $\bar{i}_*$ is the homeomorphism induced by $\bar{i}$ (cf. Section 3.2). By Lemma 3.1, r_1 is continuous. Thus, by Definition 3.4 the map r is continuous. $\square$

3.2. Induced measures

Suppose Y and Z are connected, locally connected, locally compact separable metric spaces and $f : Y \to Z$ is a proper map. For $\mu \in \mathcal{M}(Y)$ the *induced* measure $f_*\mu \in \mathcal{M}(Z)$ is defined by $(f_*\mu)(C) = \mu(f^{-1}(C))$ $(C \in \mathcal{B}(Z))$.

Lemma 3.3. *The map $f_* : \mathcal{M}(Y)_w \to \mathcal{M}(Z)_w$ is continuous.*

Suppose E is a closed subset of Y, F is a closed subset of Z, $f(E) = F$ and f maps $Y - E$ homeomorphically onto $Z - F$.

Definition 3.5. For $\nu \in \mathcal{M}^F(Z)$ we define $f^*\nu \in \mathcal{M}^E(Y)$ by $(f^*\nu)(B) = \nu(f(B - E))$ $(B \in \mathcal{B}(Y))$.

If $\nu \in \mathcal{M}_g^F(Z)$ and $\mathrm{Int}\, E = \emptyset$, then $f^*\nu \in \mathcal{M}_g^E(Y)$.

Lemma 3.4. (Cf. [3, Proposition 4.3 (2), (6)].)

(1) *The following maps are reciprocal homeomorphisms*:

$$f_* : \mathcal{M}^E(Y)_w \to \mathcal{M}^F(Z)_w, \qquad f^* : \mathcal{M}^F(Z)_w \to \mathcal{M}^E(Y)_w.$$

(2) *If $\overline{f} : \mathcal{E}(Y) \to \mathcal{E}(Z)$ is bijective, then for any $\nu \in \mathcal{M}^F(Z)$ the following maps are reciprocal homeomorphisms*:

$$f_* : \mathcal{M}^E(Y, f^*\nu\text{-}\textit{end-reg})_{ew} \to \mathcal{M}^F(Z, \nu\text{-}\textit{end-reg})_{ew},$$
$$f^* : \mathcal{M}^F(Z, \nu\text{-}\textit{end-reg})_{ew} \to \mathcal{M}^E(Y, f^*\nu\text{-}\textit{end-reg})_{ew}.$$

Proof. (1) One can check the following equalities:

$$\begin{aligned}(f^* f_*\mu)(B) &= (f_*\mu)\big(f(B-E)\big)\\ &= \mu\big(f^{-1}f(B-E)\big) = \mu(B-E) = \mu(B) \quad (B \in \mathcal{B}(Y)),\\ (f_* f^*\nu)(C) &= (f_* f^*\nu)(C-F)\\ &= (f^*\nu)\big(f^{-1}(C-F)\big) = \nu(C-F) = \nu(C) \quad (C \in \mathcal{B}(Z)).\end{aligned}$$

We omit lengthy but routine verifications of the remaining parts of (1) and (2). □

3.3. Groups of measure-preserving homeomorphisms

Suppose Y is a connected, locally connected, locally compact, separable metrizable space and $\mu \in \mathcal{M}(Y)$.

Definition 3.6. Let $h \in \mathcal{H}(Y)$. We say that

(i) *h preserves μ* if $h_*\mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(Y)$),
(ii) [5] h is *μ-biregular* if $h_*\mu$ and μ have the same null sets (i.e., $\mu(h(B)) = 0$ iff $\mu(B) = 0$ for any $B \in \mathcal{B}(Y)$),
(iii) [3] h is *μ-end-regular* if h is μ-biregular and $\alpha(h_*\mu) = \alpha(\mu)$.

Definition 3.7. Suppose A and X are closed subsets of Y.

(i) $\mathcal{H}_X(Y, A; \mu)$ denotes the subgroup of $\mathcal{H}_X(Y, A)$ consisting of μ-preserving homeomorphisms. $\mathcal{H}_X(Y, A; \mu)_0$ denotes the connected components of id_X in $\mathcal{H}_X(Y, A; \mu)$.
(ii) $\mathcal{H}_X(Y, A; \mu\text{-end-reg})$ denotes the subgroup of $\mathcal{H}_X(Y, A)$ consisting of μ-end-regular homeomorphisms. $\mathcal{H}_X(Y, A; \mu\text{-end-reg})_0$ denotes the connected components of id_X in $\mathcal{H}_X(Y, A; \mu\text{-end-reg})$.

Lemma 3.5.

(1) (Cf. [3, Section 3, p. 243].) *For $h \in \mathcal{H}(Y)$ we have*

$$\alpha(h_*\mu)(h(e)) = \alpha(\mu)(e) \quad (e \in \mathcal{E}(Y)).$$

In particular, if $h \in \mathcal{H}(Y)_0$, then $h(e) = e$ $(e \in \mathcal{E}(Y))$ and $\alpha(h_*\mu) = \alpha(\mu)$.
(2) *If $h \in \mathcal{H}_X(Y)_0$, then $h(C) = C$ for any $C \in \mathcal{C}(Y - X)$.*

Proof. (1) Note that

$$(h_*\mu)\big(h(e)\big(h(K)\big)\big) = (h_*\mu)\big(h\big(e(K)\big)\big) = \mu\big(e(K)\big) \quad (K \in \mathcal{K}(Y)). \qquad \square$$

3.4. Actions of homeomorphism groups on spaces of Radon measures

Suppose Y is a connected, locally connected, locally compact, separable metrizable space, X and A are closed subsets of Y. The topological group $\mathcal{H}(Y, A)$ acts continuously on the space $\mathcal{M}_g^A(Y)_w$ by $h \cdot \nu = h_*\nu$. For each $\nu \in \mathcal{M}_g^A(Y)_w$ the subgroup $\mathcal{H}(Y, A; \nu)$ coincides with the stabilizer $\mathcal{H}(Y, A)_\nu$ of ν under this action.

For $\mu \in \mathcal{M}_g^A(Y)$ consider the subgroup

$$\mathcal{H}_X(Y, A; \mu\text{-end-reg})' = \big\{h \in \mathcal{H}_X(Y, A; \mu\text{-end-reg}) \mid \mu\big(h(C)\big) = \mu(C)\ \big(C \in \mathcal{C}(Y - X)\big)\big\}.$$

By Lemma 3.5(2) we have $(\mathcal{H}_X(Y, A; \mu\text{-end-reg})')_0 = \mathcal{H}_X(Y, A; \mu\text{-end-reg})_0$. The above action induces the continuous action of $\mathcal{H}_X(Y, A; \mu\text{-end-reg})'$ on $\mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$. There exists a natural *orbit map*

$$\pi : \mathcal{H}_X(Y, A; \mu\text{-end-reg})' \to \mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}, \quad \pi(h) = h_*\mu.$$

A continuous *section* of the orbit map π is a map

$$\sigma : \mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew} \to \mathcal{H}_X(Y, A; \mu\text{-end-reg})_0$$

such that $\pi\sigma = \mathrm{id}$ (i.e., $\sigma(\nu)_*\mu = \nu$ ($\nu \in \mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$)). By the notation $\mathcal{S}(Y, X, A, \mu)$ we mean the existence of a section of the orbit map π for the data (Y, X, A, μ).

Lemma 3.6. (Cf. [3, Proposition 5.1 (1), (2)], [5, Corollary 3.5].) *Suppose $\mathcal{S}(Y, X, A, \mu)$ holds. Then*

(i) $(\mathcal{H}_X(Y, A; \mu\text{-}\mathit{end}\text{-}\mathit{reg})', \mathcal{H}_X(Y, A; \mu)) \cong \mathcal{H}_X(Y, A; \mu) \times (\mathcal{M}_g^A(Y, X; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_{ew}, \{\mu\})$.
(ii) $(\mathcal{H}_X(Y, A; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_0, \mathcal{H}_X(Y, A; \mu)_0) \cong \mathcal{H}_X(Y, A; \mu)_0 \times (\mathcal{M}_g^A(Y, X; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_{ew}, \{\mu\})$.
(iii) $\mathcal{H}_X(Y, A; \mu)_0$ *is a SDR of* $\mathcal{H}_X(Y, A; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_0$.

Proof. By the assumption the orbit map π has a section σ. Replacing $\sigma(\nu)$ by $\sigma(\nu)\sigma(\mu)^{-1}$, we may assume that $\sigma(\mu) = \mathrm{id}_Y$.

(i) The required homeomorphism

$$\Phi : \mathcal{H}_X(Y, A; \mu\text{-end-reg})' \cong \mathcal{H}_X(Y, A; \mu) \times \mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$$

is defined by $\Phi(h) = (\sigma(h_*\mu)^{-1}h, h_*\mu)$. The inverse is given by $\Phi^{-1}(g, \nu) = \sigma(\nu)g$.

(ii) Since $\Phi(\mathrm{id}_Y) = (\mathrm{id}_Y, \mu)$ and $\mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$ is connected, it follows that $\Phi(\mathcal{H}_X(Y, A; \mu\text{-end-reg})_0) = \mathcal{H}_X(Y, A; \mu)_0 \times \mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$.

(iii) The singleton $\{\mu\}$ is a SDR of $\mathcal{M}_g^A(Y, X; \mu\text{-end-reg})_{ew}$. $\square$

Lemma 3.7. (Cf. [3, Proof of Theorem 4.1], [5, Proof of Theorem 3.3].) *Suppose Y and Z are connected, locally connected, locally compact separable metric spaces, E is a closed subset of Y with* $\mathrm{Int}_Y E = \emptyset$ *and F is a closed subset of Z with* $\mathrm{Int}_Z F = \emptyset$. *Suppose $f : Y \to Z$ is a proper map, $f(E) = F$, f maps $Y - E$ homeomorphically onto $Z - F$ and $\bar{f} : \mathcal{E}(Y) \to \mathcal{E}(Z)$ is bijective. Let $\nu \in \mathcal{M}_g^F(Z)$. We have the induced measure $f^*\nu \in \mathcal{M}_g^E(Y)$. Under these conditions $\mathcal{S}(Y, E, E, f^*\nu)$ implies $\mathcal{S}(Z, F, F, \nu)$.*

Proof. Let $\mu = f^*\nu$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_E(Y; \mu\text{-end-reg})_0 & \xrightarrow{\pi_Y} & \mathcal{M}_g^E(Y, E; \mu\text{-end-reg})_{ew} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f_*} \\
\mathcal{H}_F(Z; \nu\text{-end-reg})_0 & \xrightarrow[\pi_Z]{} & \mathcal{M}_g^F(Z, F; \nu\text{-end-reg})_{ew}
\end{array}$$

Here, π_Y and π_Z are the orbit maps and f_* is a homeomorphism with the inverse f^* (Lemma 3.4). For each $h \in \mathcal{H}_E(Y; \mu\text{-end-reg})_0$ there exists a unique $\bar{h} \in \mathcal{H}_F(Z; \nu\text{-end-reg})_0$ with $\bar{h}f = fh$. The map φ is defined by $\varphi(h) = \bar{h}$. By the assumption the orbit map π_Y has a section σ_Y. The required section σ_Z of the orbit map π_Z is defined by $\sigma_Z = \varphi\sigma_Y f^*$. $\square$

4. Radon measures on manifolds

4.1. Section theorem—a relative version

Suppose M is a connected n-manifold. For any $\mu \in \mathcal{M}_g^\partial(M)$ the group $\mathcal{H}(M; \mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew}$.

Theorem 4.1. (von Neumann–Oxtoby–Ulam [11]) *Suppose M is a compact connected n-manifold. If $\mu, \nu \in \mathcal{M}_g^\partial(M)$ and $\mu(M) = \nu(M)$, then there exists $h \in \mathcal{H}_\partial(M)_0$ such that $h_*\mu = \nu$.*

Theorem 4.2. (A. Fathi [5], R. Berlanga [3]) *Suppose M is a connected n-manifold. Then for any $\mu \in \mathcal{M}_g^\partial(M)$ the orbit map $\pi : \mathcal{H}(M; \mu\text{-}\mathit{end}\text{-}\mathit{reg}) \to \mathcal{M}_g^\partial(M; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_{ew}$, $\pi(h) = h_*\mu$ has a section $\sigma : \mathcal{M}_g^\partial(M; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_{ew} \to \mathcal{H}_\partial(M; \mu\text{-}\mathit{end}\text{-}\mathit{reg})_0$.*

We need a relative version of this section theorem.

Corollary 4.1. *Suppose M is a connected PL n-manifold, $\mu \in \mathcal{M}_g^\partial(M)$ and X is a compact subpolyhedron of M such that $\mu(\mathrm{Fr}\, X) = 0$. Then the orbit map*

$$\pi : \mathcal{H}_X(M; \mu\text{-end-reg})' \to \mathcal{M}_g^\partial(M, X; \mu\text{-end-reg})_{ew}\colon\ \pi(h) = h_*\mu$$

has a section $\sigma : \mathcal{M}_g^\partial(M, X; \mu\text{-end-reg})_{ew} \to \mathcal{H}_{X\cup\partial}(M; \mu\text{-end-reg})_0$.

Proof. Let Y_i $(i = 1, \ldots, m)$ denote the closures of connected components of $M - X$. For each i, set $\partial Y_i = (\mathrm{Fr}_M Y_i) \cup (Y_i \cap \partial M)$ and $\mathrm{Int}\, Y_i = Y_i - \partial Y_i$. Since $\partial Y_i \subset \mathrm{Fr}_M X \cup \partial M$, we have $\mu(\partial Y_i) = 0$. By Lemma 3.2, the restriction map

$$\lambda_i : \mathcal{M}_g^\partial(M, X; \mu\text{-end-reg})_{ew} \to \mathcal{M}_g^{\partial Y_i}(Y_i; \mu|_{Y_i}\text{-end-reg})_{ew}, \quad \lambda_i(\nu) = \nu|_{Y_i}$$

is continuous.

Since the 2nd derived neighborhood of $\mathrm{Fr}_M Y_i$ in Y_i is a PL-mapping cylinder neighborhood of $\mathrm{Fr}_M Y_i$ in Y_i, we can construct a connected PL n-manifold N_i and a proper onto map $f_i : N_i \to Y_i$ such that $f_i(\partial N_i) = \partial Y_i$, f_i maps $\mathrm{Int}\, N_i$ homeomorphically onto $\mathrm{Int}\, Y_i$ and $\overline{f}_i : \mathcal{E}(N_i) \to \mathcal{E}(Y_i)$ is a homeomorphism. We apply Lemma 3.7 to these data and $\mu_i = \mu|_{Y_i} \in \mathcal{M}_g^\partial(Y_i)$. By Theorem 4.2 $\mathcal{S}(N_i, \partial, \partial, f_i^*\mu_i)$ holds, hence by Lemma 3.7 $\mathcal{S}(Y_i, \partial, \partial, \mu_i)$ also holds. Thus, we obtain a section σ_i of the orbit map

$$\pi_i : \mathcal{H}_\partial(Y_i; \mu_i\text{-end-reg})_0 \to \mathcal{M}_g^\partial(Y_i; \mu_i\text{-end-reg})_{ew}, \quad \pi_i(g) = g_*\mu_i.$$

Since $M = X \cup (\bigcup_i Y_i)$ and $\mathrm{Fr}_M X = \bigcup_i \mathrm{Fr}_M Y_i$, the required section σ of π is defined by

$$\sigma(\nu) = \begin{cases} \mathrm{id}_X & \text{on } X \\ \sigma_i(\lambda_i(\nu)) & \text{on } Y_i\ (i = 1, \ldots, m) \end{cases} \quad (\nu \in \mathcal{M}_g^\partial(M, X; \mu\text{-end-reg})). \qquad \square$$

By Corollary 4.1 and Lemma 3.6 we have the following conclusion.

Corollary 4.2. *Under the condition of Corollary 4.1, for any closed subset A of ∂M*

(i) $(\mathcal{H}_{X\cup A}(M, \mu\text{-end-reg})_0, \mathcal{H}_{X\cup A}(M; \mu)_0) \cong \mathcal{H}_{X\cup A}(M, \mu)_0 \times (\mathcal{M}_g^\partial(M, X; \mu\text{-end-reg})_{ew}, \{\mu\})$,
(ii) *$\mathcal{H}_{X\cup A}(M; \mu)_0$ is a SDR of $\mathcal{H}_{X\cup A}(M; \mu\text{-end-reg})_0$.*

4.2. PL-structures compatible with Radon measures

We show that any PL-structure can be deformed to a PL-structure compatible with a given Radon measure.

Proposition 4.1. *Suppose M is a PL n-manifold, $\mu \in \mathcal{M}_g^\partial(M)$ and $X \subset X_0$ are closed subpolyhedra of M with $\mu(\mathrm{Fr}_M X) = 0$ and $\mu(X_0 - X) = 0$. Then there exists a PL-structure on M for which*

$(\#)_1$ *X and X_0 are subpolyhedra of M, and*
$(\#)_2$ *$\mathcal{H}_X^{\mathrm{PL}}(M)_0 \subset \mathcal{H}_X(M; \mu\text{-end-reg})$.*

Proof. The PL-structure of M is given by a pair (T, φ), where T is a simplicial complex which is a combinatorial n-manifold and $\varphi : |T| \cong M$ is a homeomorphism. Since X and X_0 are subpolyhedra of M, subdividing T if necessary, we may assume that there are subcomplexes S and S_0 of T such that $X = \varphi(|S|)$ and $X_0 = \varphi(|S_0|)$. Let $T^{(i)}$ denote the i-skeleton of T, while $T_{(i)}$ denotes the set of i-simplexes of T.

Claim 1. *For each $i = 0, \ldots, n - 1$*
$(*)_i$ *there exists a PL-isotopy $f_t^i \in \mathcal{H}_{X_0\cup\partial M}^{\mathrm{PL}}(M)_0$ such that*

$$f_0^i = \mathrm{id}_M \quad \text{and} \quad \mu\big(f_1^i\varphi\big(\big|T^{(i)}\big|\big) - X\big) = 0.$$

We proceeds by the induction on i.

$(*)_0$: Since μ is a good measure, $\mu(\varphi(|T^{(0)}|)) = 0$ and we can take $f_t^0 = \mathrm{id}_M$.

$(*)_{i-1} \Rightarrow (*)_i$ ($i = 1, \ldots, n-1$): Given the isotopy f_t^{i-1} in $(*)_{i-1}$. Let $\psi = f_1^{i-1}\varphi$ and consider the barycentric subdivision $\mathrm{sd}T$ of T. For every $\sigma \in T_{(i)}$ we put $B_\sigma = \mathrm{st}(b(\sigma), \mathrm{sd}T)$ (the star of the barycenter $b(\sigma)$ of σ in $\mathrm{sd}T$). Then (i) B_σ is a PL n-ball, $\sigma \cap \partial B_\sigma = \partial\sigma$, (ii) $|T| = \bigcup_{\sigma \in T_{(i)}} B_\sigma$, $B_\sigma \cap B_\tau = \partial B_\sigma \cap \partial B_\tau$ ($\sigma, \tau \in T_{(i)}$, $\sigma \neq \tau$), (iii) $|T^{(i-1)}| \cap \mathrm{Int}\, B_\sigma = \emptyset$, $|S| \subset (\bigcup_{\sigma \in S_{(i)}} B_\sigma) \cup |T^{(i-1)}|$. The PL n-balls $\psi(B_\sigma)$ also have the similar properties.

For each $\sigma \in T_{(i)}$, (a) if $\psi(\sigma) \not\subset X_0 \cup \partial M$, then we take an isotopy $g_t^\sigma \in \mathcal{H}^{\mathrm{PL}}_{\psi(\partial B_\sigma)}(\psi(B_\sigma))$ ($t \in [0,1]$) such that $g_0^\sigma = \mathrm{id}_{\psi(B_\sigma)}$ and $\mu(g_1^\sigma\psi(\sigma)) = 0$, and (b) if $\psi(\sigma) \subset X_0 \cup \partial M$, then we put $g_t^\sigma = \mathrm{id}_{\psi(B_\sigma)}$. In (a) the isotopy g_t^σ is obtained as follows: The PL n-ball $\psi(B_\sigma)$ contains a cone C over $\psi(\sigma)$, which can be regarded as the product $\psi(\sigma) \times [0,1]$ pinched over $\partial\psi(\sigma)$ (i.e., $x \times [0,1]$ is contracted to a point for each $x \in \partial\psi(\sigma)$). There exists a level $t \in (0,1)$ with $\mu(\mathrm{Int}\,\psi(\sigma) \times \{t\}) = 0$ (otherwise, we would have $\mu(C) = \infty$). Since $\partial\psi(\sigma) = \psi(\partial\sigma) \subset \psi(|T^{(i-1)}|) - \mathrm{Int}_M X$ and $\mu(\psi(|T^{(i-1)}|) - \mathrm{Int}_M X) = 0$, it follows that $\mu(\partial\psi(\sigma)) = 0$ and so $\mu(\psi(\sigma) \times \{t\}) = 0$. The isotopy g_t^σ is constructed by sliding the base $\psi(\sigma)$ to $\psi(\sigma) \times \{t\}$ rel $\partial\psi(\sigma)$ in $\psi(B_\sigma)$.

By (ii) we can define a PL-isotopy $g_t \in \mathcal{H}^{\mathrm{PL}}(M)_0$ by $g_t = g_t^\sigma$ on $\psi(B_\sigma)$. Since $f_1^{i-1} = \mathrm{id}$ on $X_0 \cup \partial M$, in the case (a) $\psi(B_\sigma) \cap (X_0 \cup \partial M) = f_1^{i-1}(\varphi(B_\sigma) \cap (X_0 \cup \partial M)) \subset f_1^{i-1}(\varphi(\partial\sigma)) \subset f_1^{i-1}\varphi(\partial B_\sigma) = \psi(\partial B_\sigma)$. Thus we have $g_t = \mathrm{id}$ on $X_0 \cup \partial M$.

Define a PL-isotopy $f_t^i \in \mathcal{H}^{\mathrm{PL}}_{X_0 \cup \partial M}(M)_0$ by $f_t^i = f_{2t}^{i-1}$ ($t \in [0, 1/2]$) and $f_t^i = g_{2t-1} f_1^{i-1}$ ($t \in [1/2, 1]$). If $\sigma \in T_{(i)}$ and $\varphi(\sigma) \not\subset X_0 \cup \partial M$, then $\psi(\sigma) \not\subset X_0 \cup \partial M$ and $f_1^i(\varphi(\sigma)) = g_1\psi(\sigma) = g_1^\sigma\psi(\sigma)$, so $\mu(f_1^i(\varphi(\sigma))) = 0$. Since $\mu(\partial M) = 0$ and $\mu(X_0 - X) = 0$, we have $\mu(f_1^i\varphi(|T^{(i)}|) - X) = 0$. This completes the inductive step. □

Claim 2. *There exists an isotopy $h_t \in \mathcal{H}_{X_0 \cup \partial M}(M)$ such that $h_0 = \mathrm{id}_M$ and the PL-structure $\psi = h_1\varphi : |T| \cong M$ satisfies the conditions $(\#)_1$, $(\#)_2$ in Proposition* 4.1.

Proof. By Claim 1 there exists a PL-isotopy $f_t = f_t^{(n-1)} \in \mathcal{H}^{\mathrm{PL}}_{X_0 \cup \partial M}(M)_0$ such that $\mu(f_1\varphi(|T^{(n-1)}|) - X) = 0$. Since $\mu(\mathrm{Fr}\, X) = 0$, we have $\mu(f_1\varphi(|T^{(n-1)}|) - \mathrm{Int}_M X) = 0$. Since $\mu(X_0 - X) = 0$, it follows that $S_0 - S \subset T^{(n-1)}$ and $X_0 \cup \partial M \subset X \cup \varphi(|T^{(n-1)}|)$, so $X_0 \cup \partial M \subset X \cup f_1\varphi(|T^{(n-1)}|)$. For any $\sigma \in T_{(n)} - S$ we have the PL n-ball $C_\sigma = f_1\varphi(\sigma)$. Since $\partial C_\sigma = f_1\varphi(\partial\sigma) \subset f_1\varphi(|T^{(n-1)}|) - \mathrm{Int}\, X$, it follows that $\mu(\partial C_\sigma) = 0$ and $\mu_\sigma := \mu|_{C_\sigma} \in \mathcal{M}_g^\partial(C_\sigma)$.

Consider the Lebesgue measure m on $\mathbb{R}^n$. The restriction of m to the n-cube $I^n := [0,1]^n \subset \mathbb{R}^n$ is denoted by the same symbol. Since any affine isomorphism of $\mathbb{R}^n$ is m-biregular, any PL-homeomorphism between two subpolyhedra of $\mathbb{R}^n$ is also m-biregular.

Choose a PL-homeomorphism $\alpha_\sigma : C_\sigma \cong I^n$. Then $(\alpha_\sigma)_*\mu_\sigma \in \mathcal{M}_g^\partial(I^n)$ and if we set $c_\sigma = ((\alpha_\sigma)_*\mu_\sigma)(I^n)(= \mu(C_\sigma) > 0)$, then by von Neumann–Oxtoby–Ulam theorem (Theorem 4.1) there exists an isotopy $\beta_t^\sigma \in \mathcal{H}_\partial(I^n)$ such that $\beta_0^\sigma = \mathrm{id}$ and $(\beta_1^\sigma)_*(\alpha_\sigma)_*\mu_\sigma = c_\sigma m$. Thus, we have a measure-preserving homeomorphism $\gamma_\sigma = \beta_1^\sigma\alpha_\sigma : (C_\sigma, \mu_\sigma) \cong (I^n, c_\sigma m)$.

Define $g_t \in \mathcal{H}_{X \cup f_1\varphi(|T^{(n-1)}|)}(M)_0$ by $g_t|_X = \mathrm{id}_X$ and $g_t|_{C_\sigma} = \alpha_\sigma^{-1}(\beta_t^\sigma)^{-1}\alpha_\sigma \in \mathcal{H}_\partial(C_\sigma)_0$ ($\sigma \in T_{(n)} - S$). Finally we define $h_t = g_t f_t \in \mathcal{H}_{X_0 \cup \partial M}(M)_0$.

By M' we denote M with the PL-structure $\psi = h_1\varphi : |T| \cong M$. Since $\psi(\sigma) = g_1 f_1\varphi(\sigma) = g_1(C_\sigma) = C_\sigma$, the subspace C_σ forms a subpolyhedron of M'. We denote by C'_σ the PL n-ball C_σ with the triangulation $\psi : \sigma \cong C_\sigma$. It follows that $X = \psi(|S|)$, $X_0 = \psi(|S_0|)$ and $\gamma_\sigma = \beta_1^\sigma\alpha_\sigma = (\alpha_\sigma f_1\varphi)\psi^{-1} : C'_\sigma \cong I^n$ is a PL-homeomorphism.

To see $\mathcal{H}_X^{\mathrm{PL}}(M')_0 \subset \mathcal{H}_X(M; \mu\text{-end-reg})$, by Lemma 3.5(1) it suffices to show that each $h \in \mathcal{H}_X^{\mathrm{PL}}(M')$ is μ-biregular. Let $h \in \mathcal{H}_X^{\mathrm{PL}}(M')$. There exist subdivisions T_1, T_2 of T and a simplicial isomorphism $k : T_1 \to T_2$ such that $h\psi = \psi|k|$. Take any $\tau_1 \in T_1$ with $\tau_1 \not\subset |S|$. There exist $\tau_2 \in T_2$ with $|k|(\tau_1) = \tau_2$ and $\sigma_1, \sigma_2 \in T_{(n)}$ with $\tau_1 \subset \sigma_1$, $\tau_2 \subset \sigma_2$. Since $|k| = \mathrm{id}$ on $|S|$, it follows that $\tau_2 \not\subset |S|$ and $\sigma_1, \sigma_2 \not\subset |S|$ and we have the following diagram:

$$\begin{tikzcd}
C'_{\sigma_1} \arrow[d, "\gamma_{\sigma_1}"', "\cong"] & \supset & \psi(\tau_1) \arrow[r, "h"] \arrow[d, "\gamma_{\sigma_1}"', "\cong"] & \psi(\tau_2) \arrow[d, "\cong"', "\gamma_{\sigma_2}"] & \subset & C'_{\sigma_2} \arrow[d, "\cong"', "\gamma_{\sigma_2}"] \\
I^n & \supset & \gamma_{\sigma_1}(\psi(\tau_1)) \arrow[r] & \gamma_{\sigma_2}(\psi(\tau_2)) & \subset & I^n
\end{tikzcd}$$

Since $\gamma_{\sigma_2} h(\gamma_{\sigma_1})^{-1} : \gamma_{\sigma_1}(\psi(\tau_1)) \cong \gamma_{\sigma_2}(\psi(\tau_2))$ is a PL-homeomorphism, it is m-biregular. Since $\gamma_{\sigma_i} : (\psi(\tau_i), \mu_{\sigma_i}) \cong (\gamma_{\sigma_i}(\psi(\tau_i)), c_{\sigma_i} m)$ $(i = 1, 2)$ are measure-preserving, it follows that $h : \psi(\tau_1) \cong \psi(\tau_2)$ is μ-biregular. Since $h|_X = \mathrm{id}_X$ is μ-biregular, it follows that h itself is μ-biregular as required.

This completes the proof of Proposition 4.1. □

Remark 4.1. Suppose M is a PL n-manifold and $\mu \in \mathcal{M}^\partial(M)$.

(1) If $\mathcal{H}^{\mathrm{PL}}_\partial(M)_0 \subset \mathcal{H}(M; \mu\text{-end-reg})$, then $\mu(K) = 0$ for any subpolyhedron K of M with $\dim K \leqslant n-1$. Indeed, take a combinatorial triangulation T of M such that K is a subcomplex of T. Suppose $s \in T$ and $\dim s \leqslant n-1$. If $s \not\subset \partial M$, then we can find $h \in \mathcal{H}^{\mathrm{PL}}_\partial(M)_0$ such that $\mu(h(\mathrm{Int}\, s)) = 0$ (as in the proof of Claim 1 in Proposition 4.1). Since $h \in \mathcal{H}(M; \mu\text{-end-reg})$, we have $\mu(\mathrm{Int}\, s) = 0$. If $s \subset \partial M$, then $\mu(s) = 0$ since $\mu(\partial M) = 0$. This implies that $\mu(K) = 0$.

(2) If X is a closed subset of M and $\nu_0 \in \mathcal{M}(X)$, then the inclusion $i : X \subset M$ induces $i_*\nu_0 \in \mathcal{M}(M)$ and $\nu := \mu + i_*\nu_0 \in \mathcal{M}(M)$ satisfies the conditions $\nu|_X = \mu|_X + \nu_0$ and $\nu|_{M-X} = \mu|_{M-X}$. For example, if $\mu \in \mathcal{M}^\partial_g(M)$, X is a PL-arc in $\mathrm{Int}\, M$ and $\nu_0 = f_* m$ (m is the 1-dimensional Lebesgue measure on $[0, 1]$, $f : [0, 1] \cong X$ is any homeomorphism), then $\nu \in \mathcal{M}^\partial_g(M)$ and $\nu(X) > 0$.

4.3. Non-locally compactness of $\mathcal{H}_X(M, \mu)_0$

Lemma 4.1. *Suppose M is an n-manifold, X is a closed subset of M $(X \neq M)$ and $\mu \in \mathcal{M}^\partial_g(M)$. Then $\mathcal{H}_X(M, \mu)_0$ is not locally compact.*

Proof. (1) First we show that $\mathcal{H}_\partial(J^n, m)$ is not compact, where $J^n = [-1, 1]^n$ and m is the n-dimensional Lebesgue measure on $\mathbb{R}^n$. Consider the sequence of points $\mathbf{0}$, $\boldsymbol{q}_k = (0, \ldots, 0, 1/k) \in \mathrm{Int}\, J^n$ $(k \geqslant 2)$. For each $k = 2, 3, \ldots,$ we can take n-balls D_k, E_k in $\mathrm{Int}\, J^n$ (of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$) such that $\mathbf{0}$, $\boldsymbol{q}_2 \in \partial D_k$, $\mathbf{0}$, $\boldsymbol{q}_k \in \partial E_k$ and $m(D_k) = m(E_k)$, and $h_k \in \mathcal{H}_\partial(J^n)$ such that $h_k(D_k) = E_k$, $h_k(\mathbf{0}) = \mathbf{0}$ and $h_k(\boldsymbol{q}_2) = \boldsymbol{q}_k$.

For the homeomorphism $h_k : D_k \cong E_k$, since $(h_k)_* m \in \mathcal{M}^\partial_g(E_k)$ and $((h_k)_* m)(E_k) = m(D_k) = m(E_k)$, by Theorem 4.1 there exists $f_k \in \mathcal{H}_\partial(E_k)$ such that $(f_k)_*(h_k)_* m = m$. Similarly, for the homeomorphism $h_k : \mathrm{cl}(J^n - D_k) \cong \mathrm{cl}(J^n - E_k)$, since $(h_k)_* m \in \mathcal{M}^\partial_g(\mathrm{cl}(J^n - E_k))$ and $((h_k)_* m)(\mathrm{cl}(J^n - E_k)) = m(\mathrm{cl}(J^n - D_k)) = m(\mathrm{cl}(J^n - E_k))$, there exists $g_k \in \mathcal{H}_\partial(\mathrm{cl}(J^n - E_k))$ such that $(g_k)_*(h_k)_* m = m$.

Define $\varphi_k \in \mathcal{H}_\partial(J^n; m)$ by $\varphi_k = f_k h_k$ on D_k and $\varphi_k = g_k h_k$ on $\mathrm{cl}(J^n - D_k)$. Since $\|\varphi_k(\mathbf{0}) - \varphi_k(\boldsymbol{q}_2)\| = 1/k \to 0$, any subsequence of φ_k does not converge in $\mathcal{H}_\partial(J^n)$ and hence $\mathcal{H}_\partial(J^n; m)$ is not compact.

(2) Suppose $\mathcal{H}_X(M, \mu)$ is locally compact. Then id_M has a compact neighborhood $\mathcal{F}$ in $\mathcal{H}_X(M, \mu)$. There exists an $\varepsilon > 0$ such that $\mathcal{N}(\mathrm{id}_M, \varepsilon) \subset \mathcal{F}$. Take any n-ball B in M with $\mathrm{diam} B < \varepsilon$. In any collar $\partial B \times [0, 1]$ of ∂B in B, there is a level $t \in [0, 1]$ with $\mu(\partial B \times \{t\}) = 0$. Thus, replacing B by a smaller one, we may assume that $\mu(\partial B) = 0$. Put $c = \mu(B)/2^n$. As similarly to γ_σ in the proof of Proposition 4.1, there is a measure-preserving homeomorphism $(J^n, cm) \cong (B, \mu)$ and this yields a natural closed embedding $\mathcal{H}_\partial(J^n, cm) \cong \mathcal{H}_\partial(B, \mu) \hookrightarrow \mathcal{F}$ (extending by id on $M - B$). This contradicts the noncompactness of $\mathcal{H}_\partial(J^n, m)$. □

Lemma 4.2. *Suppose M is an n-manifold, X is a closed subset of M and $\mu \in \mathcal{M}^\partial_g(M)$. Then $\mathcal{H}_X(M, \mu)$ (or $\mathcal{H}_X(M, \mu)_0$) is an ℓ_2-manifold iff it is an ANR and $X \neq M$.*

Proof. Suppose $\mathcal{G} \equiv \mathcal{H}_X(M, \mu)$ (or $\mathcal{H}_X(M, \mu)_0$) is an ANR and $X \neq M$. Since $\mathcal{H}_X(M)$ is separable and completely metrizable (cf. [16]) and $\mathcal{H}_X(M, \mu)$ is a closed subgroup of $\mathcal{H}_X(M)$, it follows that $\mathcal{G}$ is also a separable, completely metrizable topological group. Since $X \neq M$, by Lemma 4.1 $\mathcal{G}$ is not locally compact. Thus, by Theorem 2.1, $\mathcal{G}$ is an ℓ_2-manifold. □

5. Groups of measure-preserving homeomorphisms of 2-manifolds

Lemma 5.1. *Suppose M is a connected 2-manifold, X is a compact subpolyhedron of M with respect to some triangulation of M, $\mu \in \mathcal{M}^\partial_g(M)$ and $\mu(\mathrm{Fr}_M X) = 0$.*

(1) $\mathcal{H}_X(M;\mu\text{-}end\text{-}reg)_0$ *is an ANR and HD in* $\mathcal{H}_X(M)_0$.
(2) $\mathcal{H}_X(M;\mu)_0$ *is an ANR and a SDR of* $\mathcal{H}_X(M;\mu\text{-}end\text{-}reg)_0$.
(3) $\mathcal{H}_X(M;\mu)_0$ *is a SDR of* $\mathcal{H}_X(M)_0$.

Proof. (1) By Proposition 4.1 M has a PL-structure such that X is a subpolyhedron and $\mathcal{H}_X^{\mathrm{PL}}(M)_0 \subset \mathcal{H}_X(M;\mu\text{-end-reg})_0$. Since $\mathcal{H}_X^{\mathrm{PL}}(M)_0 \subset \mathcal{H}_X(M;\mu\text{-end-reg})_0 \subset \mathcal{H}_X(M)_0$ and $\mathcal{H}_X^{\mathrm{PL}}(M)_0$ is HD in $\mathcal{H}_X(M)_0$ [17, Theorem 3.2], it follows that $\mathcal{H}_X(M;\mu\text{-end-reg})_0$ is also HD in $\mathcal{H}_X(M)_0$. Since $\mathcal{H}_X(M)_0$ is an ANR [16], by Lemma 2.1 $\mathcal{H}_X(M;\mu\text{-end-reg})_0$ is also an ANR.

(2) By Corollary 4.2 $\mathcal{H}_X(M;\mu)_0$ is a SDR of $\mathcal{H}_X(M;\mu\text{-end-reg})_0$. By (1) $\mathcal{H}_X(M;\mu)_0$ is also an ANR.

(3) Since $\mathcal{H}_X(M;\mu)_0$ is a closed subset of $\mathcal{H}_X(M)_0$, using the absorbing homotopy in (1) and the SDR in (2) we can easily construct a SDR of $\mathcal{H}_X(M)_0$ onto $\mathcal{H}_X(M;\mu)_0$. □

Proof of Theorem 1.1 and Corollary 1.1. The assertions follow from Lemmas 5.1 and 4.2. □

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